Kato Theorem Part 3

Ege Şirin

Intro

Definition

Definition

- $A_i : L(\mathbf{r_i})^2 \to L(\mathbf{r_i})^2$
- $A_i f(\mathbf{r}_i) = \left(\int \mid f(\mathbf{r}) \mid^2 d\mathbf{r_1} \cdots d\mathbf{r_{i-1}} d\mathbf{r_{i+1}} \cdots d\mathbf{r_s}\right)^{1/2}$

Properties

For all $i = 1 \cdots s$,

- $\|A_i f\| = \|f\|$
- $\mid (A_i f - A_i g)(\mathbf{r_i}) \mid \leq A_i (f - g)(\mathbf{r_i})$
- $\|A_i f - A_i g\| \leq \|f - g\|$

Lemma
If $f(\mathbf{r}) \in \mathscr{D}_0$ then for all $i = 1 \cdots$, $A_i f$ are continuous and bounded as

$$0 \leq A_i f(\mathbf{r_i}) \leq a' \|T_0 f\| + b' \|f\| \tag{1}$$

where a' and b' are constants independent of f, and a' can be taken as small as we want.

strategy of the proof

1) for any $g \in \mathscr{D}_1$, show that the ineqaulity 1 holds
2) extend domain to $\mathscr{D}_0$ such that the inequality 1 holds for all $f \in \mathscr{D}_0$

Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\mathbf{r}_i) \leq a' \|T_0 f\| + b' \|g\|$ holds.

It is enough to show that the inequality holds for $i = 1$.

$$\|A_1 g(\mathbf{r}_1)\|^2 = \int \|g(\mathbf{r})\|^2 d\mathbf{r}_2 \cdots \mathbf{r}_s$$

$$g(\mathbf{r}) = (2\pi)^{-3s/2} \int \exp(\{i(\mathbf{p}_2\mathbf{r}_2 + \cdots + \mathbf{p}_s\mathbf{r}_s)\} d\mathbf{p}_2 \cdots \mathbf{p}_s \cdot \int \exp(i\mathbf{p}_1\mathbf{r}_1) G(\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_2) d\mathbf{p}_1$$

Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\mathbf{r_i}) \leq a' \|T_0 f\| + b' \|g\|$ holds.

It is enough to show that the inequality holds for $i = 1$.

$$
\begin{aligned}
\|A_1 g(\mathbf{r_1})\|^2 &= \int \|g(\mathbf{r})\|^2 d\mathbf{r_2} \cdots \mathbf{r_s} \\
&= (2\pi)^{-3} \int d\mathbf{p_2} \cdots d\mathbf{p_s} \left| \int \exp(i\mathbf{p_1}\mathbf{r_1}) G(\mathbf{p}) d\mathbf{p_1} \right|^2 \\
&\leq (2\pi)^{-3} \int d\mathbf{p_2} \cdots d\mathbf{p_s} \left\{ \int |G(\mathbf{p})| d\mathbf{p_1} \right\}^2
\end{aligned}
$$

- Parseval identity applied to $3(s-1)$ variables $\mathbf{r_2}, \cdots \mathbf{r_s}$
- $\left| \int f \right| \leq \int |f|$

Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\mathbf{r_i}) \leq a' \|T_0 g\| + b' \|g\|$ holds.

It is enough to show that the inequality holds for $i = 1$.

$$\begin{aligned}
\|A_1 g(\mathbf{r_1})\|^2 &= \int \|g(\mathbf{r})\|^2 d\mathbf{r_2} \cdots \mathbf{r_s} \\
&= (2\pi)^{-3} \int d\mathbf{p_2} \cdots d\mathbf{p_s} \Big| \int \exp(i\mathbf{p_1}\mathbf{r_1}) G(\mathbf{p}) d\mathbf{p_1} \Big|^2 \\
&\leq (2\pi)^{-3} \int d\mathbf{p_2} \cdots d\mathbf{p_s} \Big\{ \int |G(\mathbf{p})| d\mathbf{p_1} \Big\}^2 \\
&\leq (2\pi)^{-3} \int d\mathbf{p_2} \cdots d\mathbf{p_s} \int |G(\mathbf{p})|^2 (1 + k^4 \mathbf{p_1}^4) d\mathbf{p_1} \cdot \\
&\quad \int (1 + k^4 \mathbf{p_1}^4)^{-1} d\mathbf{p_1}
\end{aligned}$$

- $\Big\{ \int |G(\mathbf{p})| d\mathbf{p_1} \Big\}^2 \leq \int |G(\mathbf{p})|^2 (1 + k^4 \mathbf{p_1}^4) d\mathbf{p_1} \cdot \int (1 + k^4 \mathbf{p_1}^4)^{-1} d\mathbf{p_1}$
  where $k > 0$ constant.

Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\boldsymbol{r_i}) \leq a' \| T_0 g \| + b' \| g \|$ holds.

It is enough to show that the inequality holds for $i = 1$.

$$
\begin{aligned}
\| A_1 g(\boldsymbol{r_1}) \|^2 &= \int \| g(\boldsymbol{r}) \|^2 d\boldsymbol{r_2} \cdots \boldsymbol{r_s} \\
&= (2\pi)^{-3} \int d\boldsymbol{p_2} \cdots d\boldsymbol{p_s} \left| \int \exp(i\boldsymbol{p_1}\boldsymbol{r_1}) G(\boldsymbol{p}) d\boldsymbol{p_1} \right|^2 \\
&\leq (2\pi)^{-3} \int d\boldsymbol{p_2} \cdots d\boldsymbol{p_s} \left\{ \int |G(\boldsymbol{p})| d\boldsymbol{p_1} \right\}^2 \\
&\leq (2\pi)^{-3} \int d\boldsymbol{p_2} \cdots d\boldsymbol{p_s} \int |G(\boldsymbol{p})|^2 (1 + k^4 \boldsymbol{p_1}^4) d\boldsymbol{p_1} \cdot \\
& \int (1 + k^4 \boldsymbol{p_1}^4)^{-1} d\boldsymbol{p_1} \\
&= (2\pi)^{-3} c k^{-3} \int |G(\boldsymbol{p})|^2 (1 + k^4 \boldsymbol{p_1}^4) d\boldsymbol{p_1} d\boldsymbol{p_2} \cdots d\boldsymbol{p_s} \cdot
\end{aligned}
$$

- $\int (1 + k^4 \boldsymbol{p_1}^4)^{-1} d\boldsymbol{p_1} = c k^{-3}$ for some constant c

Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\boldsymbol{r}_i) \leq a' \|T_0 g\| + b' \|g\|$ holds.

It is enough to show that the inequality holds for $i = 1$.

$$\|A_1 g(\boldsymbol{r}_1)\|^2 = \int \|g(\boldsymbol{r})\|^2 d\boldsymbol{r}_2 \cdots \boldsymbol{r}_s$$

$$= (2\pi)^{-3} \int d\boldsymbol{p}_2 \cdots d\boldsymbol{p}_s \left| \int \exp(i\boldsymbol{p}_1 \boldsymbol{r}_1) G(\boldsymbol{p}) d\boldsymbol{p}_1 \right|^2$$

$$\leq (2\pi)^{-3} \int d\boldsymbol{p}_2 \cdots d\boldsymbol{p}_s \left\{ \int |G(\boldsymbol{p})| d\boldsymbol{p}_1 \right\}^2$$

$$\leq (2\pi)^{-3} \int d\boldsymbol{p}_2 \cdots d\boldsymbol{p}_s \int \left|G(\boldsymbol{p})\right|^2 (1 + k^4 {\boldsymbol{p}_1}^4) d\boldsymbol{p}_1 \cdot$$

$$\int (1 + k^4 {\boldsymbol{p}_1}^4)^{-1} d\boldsymbol{p}_1$$

$$= (2\pi)^{-3} c k^{-3} \int \left|G(\boldsymbol{p})\right|^2 (1 + k^4 {\boldsymbol{p}_1}^4) d\boldsymbol{p}_1 d\boldsymbol{p}_2 \cdots d\boldsymbol{p}_s \cdot$$

$$= (2\pi)^{-3} c k^{-3} \left(\|G\|^2 + k^4 \|{\boldsymbol{p}_1}^2 G\|^2 \right)$$

Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\mathbf{r_i}) \leq a' \|T_0 g\| + b' \|g\|$ holds. It is enough to show that the inequality holds for $i = 1$.

$$\begin{aligned} \|A_1 g(\mathbf{r_1})\|^2 &\leq (2\pi)^{-3} c k^{-3} \left(\|G\|^2 + k^4 \|\mathbf{p_1}^2 G\|^2 \right) \\ &\leq (2\pi)^{-3} c \left(k \mu_1^{-2} \|T_0 g\|^2 + k^{-3} \|g\|^2 \right) \end{aligned}$$

- $\mu_1 \|\mathbf{p_1}^2 G\| \leq \|T_0 G\|$
- $g \rightleftarrows G$

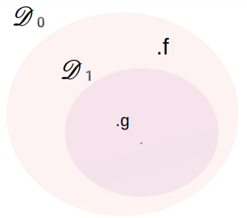
Proof

1) For any $g \in \mathscr{D}_1$, we show $0 \leq A_i g(\mathbf{r_i}) \leq a' \|T_0 g\| + b' \|g\|$ holds. It is enough to show that the inequality holds for $i = 1$.

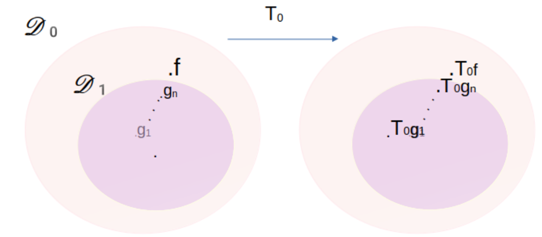
$$\begin{aligned} \|A_1 g(\mathbf{r_1})\|^2 &\leq (2\pi)^{-3} c k^{-3} \left(\|G\|^2 + k^4 \|\mathbf{p_1}^2 G\|^2\right) \\ &\leq (2\pi)^{-3} c \left(k \mu_1^{-2} \|T_0 g\|^2 + k^{-3} \|g\|^2\right) \\ &\leq a |T_0 g\|^2 + b \|g\|^2 \end{aligned}$$

$\Rightarrow 0 \leq A_1 g(\mathbf{r_1}) \leq a' |T_0 g\| + b' \|g\|$

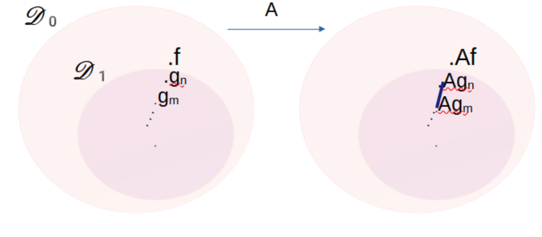
2)extend domain to $\mathscr{D}_0$ such that the inequality 1 holds for all $f \in \mathscr{D}_0$



Proof

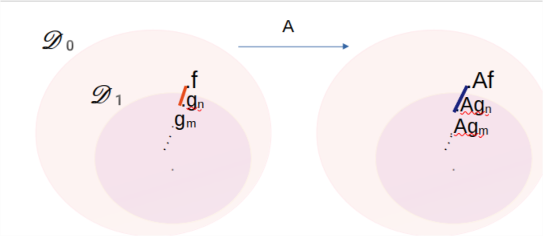


$$\|g_n - f\| \to 0, \|T_0 g_n - T_0 f\| \to 0 \text{ as } n \to \infty$$



$$|Ag_n(r_1) - Ag_m(r_1)| \le A_1(g_n - g_m)(r_1) \le a'\|T_0 g_n - T_0 g_m\| + b'\|g_n - g_m\| \to 0$$

Proof



$$|A_1 g_n - A_1 f\| \leq \|g_n - f\| \to 0$$

There is a subsequence of $(A_1 g_n)_{n \in \mathcal{N}} \to A_1 f$

We know $0 \leq A_1 g_n(\mathbf{r_1}) \leq a'|T_0 g_n\| + b'\|g_n\|$

$0 \leq A_1 f(\mathbf{r_1}) \leq a'|T_0 f\| + b'\|f\|$

Potential Energy

Introduction to Potential Energy

- Our main goal $H = T + V$ is self adjoint
- Kinetic Energy T (last week)
- Potential Energy V (today)

Potential Energy

Potential Energy

- $V : Dom(V) \subset \mathcal{L}^2 \to \mathcal{L}^2, \qquad \mathcal{L} = \mathcal{L}^2(\mathcal{R}^{3s})$
- $V(\mathbf{r}_1, \cdots \mathbf{r}_s) = V'(\mathbf{r}_1, \cdots \mathbf{r}_s) + \sum_{i=1}^{s} V_{0i}(\mathbf{r}_i) + \sum_{i<j}^{1,s} V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

Assumptions

- $\|V'(\mathbf{r}_1, \cdots \mathbf{r}_s)\| \leq C$
- $\int_{\mathbf{r} \leq R} \|V_{ij}(x, y, z)\|^2 dxdydz \leq C^2$
- $\|V_{ij}(x, y, z)\| \leq C \quad (r > R)$

Potential Energy

- $V : Dom(V) \subset \mathcal{L}^2 \to \mathcal{L}^2, \qquad \mathcal{L} = \mathcal{L}^2(\mathcal{R}^{3s})$
- $V(\mathbf{r}_1, \cdots \mathbf{r}_s) = V'(\mathbf{r}_1, \cdots \mathbf{r}_s) + \sum_{i=1}^{s} V_{0i}(\mathbf{r}_i) + \sum_{i<j}^{1,s} V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

Properties

- $V$is real mutiplicative operator

$$V_t : Dom(V) \to \mathcal{L}^2$$
$$f(\mathbf{r}) \mapsto t(\mathbf{r}) \cdot f(\mathbf{r})$$

- V is symmetric

$$\langle Vf, g\rangle = \int t(\mathbf{r})f(\mathbf{r})g(\mathbf{r})dr = \int f(\mathbf{r})t(\mathbf{r})g(\mathbf{r})dr = \langle f, Vg\rangle$$

- V is self adjoint.

Lemma

$Dom(V)$ contains $\mathcal{D}_0$ and for all $f \in \mathcal{D}_0$ there two constants a and b such that

$$\|Vf\| \leq a\|T_0 f\| + b\|f\| \tag{2}$$

Moreover, a can be taken as small as we want.

Proof

$V(\mathbf{r}_1, \cdots \mathbf{r}_s) = V'(\mathbf{r}_1, \cdots \mathbf{r}_s) + \sum_{i=1}^{s} V_{0i}(\mathbf{r}_i) + \sum_{i<j}^{1,s} V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

1) $V = V'(\mathbf{r}_i)$
2) $V = V_{0i}(\mathbf{r})$
3) $V = V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

1) We know $\|V'(\mathbf{r}_1, \cdots \mathbf{r}_s)\| \leq C$, take $a = 0$, and $b = C$.
$\|Vf\| \leq \|V\|\|f\| \leq C\|f\| = 0 \cdot \|T_0 f\| + C\|f\|$

Lemma

Proof

✓ $V = V'(\boldsymbol{r}_i)$

2) $V = V_{0i}(\boldsymbol{r})$

$$
\begin{aligned}
\|V_{01}f\|^2 &= \int |V_{01}(\boldsymbol{r}_1)|^2 |f(\boldsymbol{r}_1 \cdots \boldsymbol{r}_s)|^2 d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_s \\
&= \int |V_{01}(\boldsymbol{r}_1)|^2 |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 \\
&= \int_{r_1 \le R} |V_{01}(\boldsymbol{r}_1)|^2 |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 + \int_{r_1 \ge R} |V_{01}(\boldsymbol{r}_1)|^2 |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 \\
&\le (2a'^2 \|T_0 f\| + 2b'^2 \|f\|^2) \int_{r_1 \le R} |V_{01}(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 + C^2 \int_{r_1 \ge R} |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1
\end{aligned}
$$

$|A_1 f(\boldsymbol{r}_1)|^2 \le 2a'^2 \|T_0 f\| + 2b'^2 \|f\|^2 \qquad \|V_{ij}(x, y, z)\| \le C \quad (r > R)$

Proof

✓ $V = V'(\boldsymbol{r}_i)$

2) $V = V_{0i}(\boldsymbol{r})$

$$\begin{aligned}
\|V_{01}f\|^2 &= \int |V_{01}(\boldsymbol{r}_1)|^2 |f(\boldsymbol{r}_1 \cdots \boldsymbol{r}_s)|^2 d\boldsymbol{r}_1 \cdots d\boldsymbol{r}_s \\
&= \int |V_{01}(\boldsymbol{r}_1)|^2 |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 \\
&= \int_{r_1 \leq R} |V_{01}(\boldsymbol{r}_1)|^2 |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 + \int_{r_1 \geq R} |V_{01}(\boldsymbol{r}_1)|^2 |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 \\
&\leq (2a'^2 \|T_0 f\| + 2b'^2 \|f\|^2) \int_{r_1 \leq R} |V_{01}(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 + C^2 \int_{r_1 \geq R} |A_1 f(\boldsymbol{r}_1)|^2 d\boldsymbol{r}_1 \\
&\leq C^2 (2a'^2 \|T_0 f\| + 2b'^2 \|f\|^2) + C^2 \|f\|^2
\end{aligned}$$

$\int_{\boldsymbol{r} \leq R} \|V_{ij}(x, y, z)\|^2 dxdydz \leq C^2$ $\qquad \|A_i f\| = \|f\|$

Lemma

Proof

- ✓ $V = V'(\mathbf{r}_i)$
- ✓ $V = V_{0i}(\mathbf{r})$

3) $V = V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

$$\mathbf{r}_1{}' = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{r}_2{}' = \mathbf{r}_2, \cdots, \mathbf{r}_s{}' = \mathbf{r}_s$$

$$d\mathbf{r}_1 \cdots d\mathbf{r}_s = d\mathbf{r}_1{}' \cdots d\mathbf{r}_s{}'$$

$$f(\mathbf{r}_1 \cdots \mathbf{r}_s) = f'(\mathbf{r}_1{}' \cdots \mathbf{r}_s{}')$$

Lemma

Proof

- ✓ $V = V'(\mathbf{r}_i)$
- ✓ $V = V_{0i}(\mathbf{r})$
- 3) $V = V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

$$\mathbf{r}_1{}' = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{r}_2{}' = \mathbf{r}_2, \cdots, \mathbf{r}_s{}' = \mathbf{r}_s$$

$$d\mathbf{r}_1 \cdots d\mathbf{r}_s = d\mathbf{r}_1{}' \cdots d\mathbf{r}_s{}'$$

$$f(\mathbf{r}_1 \cdots \mathbf{r}_s) = f'(\mathbf{r}_1{}' \cdots \mathbf{r}_s{}')$$

$\|V_{12}f\|^2 = \int |V_{12}(\mathbf{r}_1{}')|^2 |f'(\mathbf{r}_1{}', \cdots, \mathbf{r}_s)|^2 d\mathbf{r}_1{}' \cdots d\mathbf{r}_s{}'$

Lemma

Proof

- $V = V'(\mathbf{r}_i)$
- $V = V_{0i}(\mathbf{r})$
- $V = V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$

$$\mathbf{r}_1{}' = \mathbf{r}_1 - \mathbf{r}_2, \quad \mathbf{r}_2{}' = \mathbf{r}_2, \cdots, \mathbf{r}_s{}' = \mathbf{r}_s$$

$$d\mathbf{r}_1 \cdots d\mathbf{r}_s = d\mathbf{r}_1{}' \cdots d\mathbf{r}_s{}'$$

$$f(\mathbf{r}_1 \cdots \mathbf{r}_s) = f'(\mathbf{r}_1{}' \cdots \mathbf{r}_s{}')$$

$$\|V_{12}f\|^2 = \int |V_{12}(\mathbf{r}_1{}')|^2 |f'(\mathbf{r}_1{}', \cdots, \mathbf{r}_s)|^2 d\mathbf{r}_1{}' \cdots d\mathbf{r}_s{}'$$