Kato Theorem Part 4

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Motivation

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- This week we'll show $H = T + V$ is self adjoint

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We know

- T is self-adjoint
- V is self-adjoint

Can we say $T + V$ is self-adjoint?

Motivation

Generally, if

- T_1 is self-adjoint
- T_2 is self-adjoint

Does it imply that $T_1 + T_2$ is self-adjoint?

Motivation

- T_1 is defined $\mathcal{D}(T_1)$
- T_2 is defined $\mathcal{D}(T_2)$

$T_1 + T_2$ is defined on $\mathcal{D}(T_1 + T_2) = \mathcal{D}(T_1) \cap \mathcal{D}(T_2)$

Motivation

There are two ways to show $H = T + V$ is self adjoint

1 Kato-Rellich Theorem

✓ Direct proof

Intro

Definition

Let $X \neq 0$ be a complex normed space and $T : \mathcal{D}(T) \to X$ is a linear operator and $\mathcal{D}(T) \subset X$. With T we associate the operator

$$T_\lambda = T - \lambda I$$

where $\lambda \in \mathbb{C}$ and I is the identity operator on $\mathcal{D}(T)$. If T_λ has an inverse, we call ***resolvent*** operator of T

$$R_\lambda(T) := T_\lambda^{-1} = (T - \lambda I)^{-1}$$

$R_\lambda(T)$ helps to solve the equation $T_\lambda x = y$. Thus, $x = T_\lambda^{-1} y = R_\lambda(T) y$ provided $R_\lambda(T)$ exists.

Regular Value, Resolvent Set, Spectrum

$T : \mathcal{D}(T) \to X$ be a linear operator. A ***regularvalue*** λ of T

(R1) $R_\lambda(T)$ exists,

(R2) $R_\lambda(T)$ is bounded,

(R3) $R_\lambda(T)$ is defined on a set which is dense in X.

The ***resolvent set*** $\rho(T)$ of T is the set of all refular values of λ of T.
$\sigma(T) = \mathsf{C} - \rho(T)$ is called the ***spectrum*** of T.

Thm: $\lambda \in \rho(T)$ iff there exists a $c > 0$ such that for all $x \in \mathcal{D}(T)$,

$$c\|x\| \leq \|T_\lambda x\|$$

Point Spectrum, Continuous Spectrum, Residual Spectrum

Satisfied	Not Satisfied	λ belongs to
R1, R2,R3	-	$\rho(T)$
-	R1	$\sigma_p(T)$
R1,R3	R2	$\sigma_c(T)$
R1	R3	$\sigma_r(T)$

$$\begin{aligned} \mathsf{C} &= \rho(T) \cup \sigma(T) \\ &= \sigma_p(T) \cup \sigma_c(T) \cup \sigma_d(T) \end{aligned}$$

If $T_\lambda x = (T - \lambda I)x = 0$ for some $x \neq 0$ then $\lambda \in \sigma_p(T)$, by definition is eigenvalue of T. The vector x is then called eigenvector (or eigenfunction -if X is function space) of T.

Domain of $R_\lambda(T)$

Lemma 1Let $T : X \to X$ be a linear operor, and $\lambda \in \rho(T)$. If T is closed or bounded then $R_\lambda(T)$ is defined on the whole space X and is bounded.

Proof Since T is closed, so is T_λ. Hence R_λ is closed. R_λ bounded by R2. Hence $\mathcal{D}(R_\lambda)$ is closed, R3 implies that $\mathcal{D}(R_\lambda) = \overline{\mathcal{D}(R_\lambda)} = X$
Since $\mathcal{D}(T) = X$ and T is bounded, T is closed (the remaning of the proof is the same as before).

Spectrum

Theorem The spectrum $\sigma(T)$ of a self-adjoint operator $T : \mathcal{D}(T) \to H$ is real and closed, here H is complex Hilbert space and $\mathcal{D}(T)$ dense in H.
Proof For all $x \neq 0$ in $\mathcal{D}(T)$ we have

$$\langle T_\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle$$

and since $\langle x, x\rangle$, $\langle Tx, x\rangle$ are real

$$\overline{\langle T_\lambda x, x\rangle} = \langle Tx, x\rangle - \overline{\lambda}\langle x, x\rangle$$

$$2iIm\langle T_\lambda x, x\rangle = \overline{\langle T_\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = (\lambda - \overline{\lambda})\langle x, x\rangle = 2i\beta\|x\|^2$$

By Schwatz inequality,

$$|\beta|\|x\|^2 \leq |\langle T_\lambda x, x\rangle| \leq \|T_\lambda x\|\|x\|$$

we have, $|\beta|\|x\| \leq \|T_\lambda x\|$ for all x. If λ is not real, $\beta \neq 0$, so that $\lambda \in \rho(T)$ by previous theorem. Hence $\sigma(T)$ must be real.

Inverse

Lemma 2 $T : X \to X$. If $\|T\| < 1$ then $(I - T)^{-1}$ exists as a bounded linear operator on the whole space X and

$$(I - T)^{-1} = \sum_{j=0}^{\infty} T^j = I + T + T^2 + \cdots$$

Proof We have $\|T^j\| \le \|T\|^j$. Note that $\sum_{j=0}^{\infty} T^j$ converges for $\|T\| < 1$. Let S be the sum of the series. It remains to show that $S = (I - T)^{-1}$

$$(I - T)(I + T + \cdots T^n) = (I + T + \cdots T^n)(I - T) = I - T^{n+1}$$

Since $\|T\| < 1$, $T^{n+1} \to 0$ when $n \to \infty$. Thus

$$(I - T)S = S(I - T) = I$$

This shows $S = (I - T)^{-1}$

Spectral Representation of Self-Adjoint Linear Operators

Let $T : \mathcal{D} \to H$ be an operator on Hilbert space H, where $\mathcal{D}$ is dense in H and T may be unbounded. The operator U

$$U = (T - iI)(T + iI)^{-1}$$

is called *Cayley transform* of T[kreyszig1991introductory].

Lemma 3 The Cayley Transform of a self-adjoint operator $T : \mathcal{D}(T) \to H$ exists and unitary operator.

Proof Since T is self adjoint, $\sigma(T)$ is real. Hence $i, -i \in \rho(T)$. Consequently, by definition of $\rho(T)$, the inverses $(T + iI)^{-1}$ and $(T - iI)^{-1}$ exists on a dense subset of of H and are bounded operators. Since T^* is closed and $T = T^*$, T is closed. By Lemma 1,

$$\mathcal{R}(T + iI) = H \quad \mathcal{R}(T - iI) = H$$

We thus have, since I is defined on all of H,

$$(T + iI)^{-1}(H) = \mathcal{D}(T + iI) = \mathcal{D}(T) = \mathcal{D}(T - iI)$$

as well as

$$(T + iI)^{-1}(\mathcal{D}(T)) = H$$

Cayley Transform

It remains to show U is isometric. Let $x \in H$ and $y = (T + iI)^{-1}x$

$$\begin{aligned}
\|Ux\|^2 &= \|(T - iI)y\|^2 \\
&= \langle Ty - iy, Ty - iy \rangle \\
&= \langle Ty, Ty \rangle + i\langle Ty, y \rangle - i\langle y, Ty \rangle + \langle iy, iy \rangle \\
&= \langle Ty + iy, Ty + iy \rangle \\
&= \|(T + iI)y\|^2 \\
&= \|(T + iI)(T + iI)^{-1}x\|^2 \\
&= \|x\|^2
\end{aligned}$$

Lemma

Lemma The operator $T_0 + V$ defined on $\mathcal{D}_0$ is self-adjoint.[10.2307/1990366]

Proof

- $\mathcal{R}(T_0 + V) = \mathcal{R}(T_0 + V + \lambda I) = \lambda \mathcal{R}(T_0 + V + iI)$
- $T_0 + V : \mathcal{D}(T_0 + V) \to \mathcal{R}(T_0 + V)$ is self adjoint iff $\mathcal{R}(T_0 + V + iI) = L^2$
- $$\begin{aligned}(T_0 + V + \lambda I) &= (T_0 + \lambda I) + V \\ &= (T_0 + \lambda I) + V(T_0 + \lambda I)(T_0 + \lambda I)^{-1} \\ &= (1 + V(T_0 + \lambda I)^{-1})(T_0 + \lambda I) \\ &= (1 + VR_\lambda(T_0))T_{0\lambda}\end{aligned}$$

*Since $T_0 : \mathcal{D}_0 \to L^2$ is self-adjoint, $T_{0\lambda} : \mathcal{D}_0 \to L^2$ by definition.
*By lemma 1, $R_\lambda(T_0)$ exists and defined on L^2.

Lemma

Let $\phi \in L^2$, then we have

$$\begin{aligned} \|VR_\lambda(T_0)\phi\| &\leq a\|T_0 R_\lambda(T_0)\phi\| + b\|R_\lambda(T_0)\phi\| && (1) \\ &\leq a\|\phi\| + b\lambda^{-1}\|\phi\| && (2) \\ &= a + b\lambda^{-1} && (3) \\ &< 1 && (4) \end{aligned}$$

Therefore, $(1 + VR_\lambda(T_0))^{-1}$ exists and defined on L^2 (Lemma 2). Hence, $\mathcal{R}(1 + VR_\lambda(T_0)) = L^2$

Theorem

Theorem H_1 is essentially self-adjoint, and its unique self-adjoint extension coincides with $T_0 + V$

Proof We know

- ✓ Domain of H_1 is $\mathcal{D}_1$ and $\mathcal{D}_1 \subseteq \mathcal{D}_0$
- ✓ $H_1 = T_1 + V \subseteq T_0 + V$
- ✓ V is relatively bounded by T_0 (i.e $\|Vf\| \leq a\|T_0 f\| + b\|f\|$)
- ✓ $\tilde{T}_1 = T_0$, T_1 is essentially self-adjoint.

Therefore, H_1 is essentially self-adjoint.

References

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Introductory Functional Analysis with Applications
Wiley Classics Library, (1991).